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On over-reflection and generation of Gravito-Alfvén waves in solar-type stars

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ABSTRACT

The dynamics of linear perturbations is studied in magnetized plasma shear flows with a constant shearing rate and with gravity-induced stratification. The general set of linearized equations is derived and the two-dimensional case is considered in detail. The *Boussinesq approximation* is used in order to examine relatively small-scale perturbations of low-frequency modes: *Gravito-Alfvén waves* (GAW) and *Entropy Mode* (EM) perturbations. It is shown that for flows with arbitrary shearing rate there exists a finite time interval of *non-adiabatic* evolution of the perturbations. The non-adiabatic behavior manifests itself in a twofold way, viz. by the over-reflection of the GAWs and by the generation of GAWs from EM perturbations. It is shown that these phenomena act as efficient transformers of the equilibrium flow energy into the energy of the perturbations

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for moderate and high shearing rate solar plasma flows. Efficient generation of GAW by EM takes place for shearing rates about an order of magnitude smaller than necessary for development of a shear instability. The latter fact could have important consequences for the problem of angular momentum redistribution within the Sun and solar-type stars.

Subject headings: MHD — Sun: magnetic fields — Sun: rotation — waves — stars: rotation

1. Introduction

It is well-known that the most advanced stellar models, taking into account rotation-induced hydrodynamic processes of meridional circulation and shear mixing, are quite adequately describing the structure of massive stars (Maeder & Meynet 2000). For solar-type stars with relatively wide convective regions, however, these models predict *large* rotation gradients and are in a notable contradiction with recent observational helioseismology data (Charbonnel & Talon 2005). This is probably due to the insufficiently high efficiency of the common hydrodynamic instabilities that are characteristic to these models: these instabilities can redistribute the angular momentum between the different layers of the radiative region, but they are *not* strong enough to ensure the onset of a uniform rotation regime.

While the radiative interiors of solar-type stars rotate quite uniformly, their convective zones rotate differentially. Mounting observational evidence indicates that the radiative interior of the Sun, for instance, rotates almost as a solid body, with a quasi-flat seismic rotation profile (Chaplin et al. 1999; Couvidat et al. 2003). But the solar convective zone exhibits a strong shearing in the latitudinal direction, with its equatorial layers rotating about 1/3 faster than the polar regions (Kim & MacGregor 2003). The transition between the differential and the uniform rotation regimes takes place in a relatively thin layer (with thickness $\leq 0.05R_{\odot}$), called the *tachocline*, where the radial shearing of the rotation is particularly significant (Miesch 2005; Petrovay 2003). It is often argued that some kind of shear instability must be the source of gravity waves at the base of the solar convection zone (Kumar, Talon & Zahn 1999), either in the immediate vicinity or just inside the tachocline. This instability can hardly be of a purely hydrodynamical nature, because from various numerical studies of the buoyant rise of magnetic flux from the bottom of the convection zone to the surface of the sun it follows that the tachocline is strongly magnetized with its poloidal magnetic field about $10^4 - 10^5$ G (Miesch 2005).

From this perspective it seems quite plausible to surmise that on the time-scale of the

order of the solar age or less not only purely hydrodynamic internal gravity waves (IGWs) participate in the process of the angular momentum redistribution within the solar interior, but also, or rather, Gravito-MHD waves. This idea is *not* new. In the literature it has been argued (Zahn, Talon & Matias 1997; Kumar & Quataert 1997) that Gravito-MHD waves could be one of the most promising candidates for the onset of the redistribution processes. It was argued that a self-consistent model should comprise a large-scale magnetic field in the Sun's interior, and an accurate consideration of the Coriolis effects in the convection zone and in the tachocline (Talon & Charbonnel 1998, 2003, 2004, 2005; Charbonnel & Talon 2005). It was also argued that turbulent stresses in the convection zone, via the agency of Coriolis effects, induce a meridional circulation, causing the gas from the convection zone to burrow downwards and generate both horizontal and vertical velocity shear characterizing the tachocline. However, the interior magnetic field confines the burrowing and, hence, smooths and diminishes the shear, as demanded by the observed structure of the tachocline. According to Charbonnel & Talon (2005), a decisive test of this *qualitative* scenario, after numerical refinements, would be its *quantitative* consistency with the observed interior angular velocity structure.

The aim of the present paper is to present the results of a detailed study of the linear dynamics of perturbations in gravitationally stratified magnetized shear flows. Bearing in mind the highly probable importance of the Gravito-MHD waves for the structure of the Sun and solar-type stars, we decided to pose and study the following three interrelated issues: 1) how the presence of the nonuniform velocity field affects the propagation of the waves through the stellar plasma; 2) what kind of energy exchange processes between the different collective modes and between the modes and the ambient flow may happen; and 3) what other astrophysical consequences these processes could have.

Originally, we develop the three-dimensional (3D) model, allowing the ambient flow to have velocity shearing in both transversal (prior to the gravity field) directions. Further on, the study is focused by a number of simplifying assumptions, enabling the solution of the equations. First of all, we consider only two spatial dimensions (2D) bearing in mind that it is quite straightforward to extend the analysis to the fully three-dimensional case. Second, we assume a constant steady-state (or stationary) shear flow, an assumption that allows us to use the *shearing sheet approximation* (Goldreich & Lynden-Bell 1965). And third, we use the *Boussinesq approximation*, and hence we study the dynamics of relatively small-scale[1] perturbations of the low-frequency modes, viz. the Gravito-Alfvén waves (GAW) and the Entropy Mode (EM) perturbations.

[1]with the vertical wavelength, ℓ_z, much smaller then stratification length scale z_0.

Our study leads to the conclusion that for *arbitrary* shear in the flow, there exists a finite interval of time in which the evolution of the modes is highly non-adiabatic. Outside this interval, the evolution is adiabatic and can be accurately described by a WKB approximation. Therefore, we have an asymptotic problem, quite common for various quantum-mechanical applications, where one needs to obtain connection formulae in *turning point problems* (Landau & Lifshitz 1977). The relevant analysis is done in this paper and it is found that the non-adiabatic behavior of the considered modes manifests itself in the form of two phenomena, viz. the *over-reflection* of the GAW and the *generation* of the GAW by the EM perturbations. We show that for flows with moderate and high shearing rates both these processes are very effective converters of the equilibrium flow energy into the energy of the waves.

In the context of above-discussed astrophysical problems our results imply that efficient generation of the GAW by the EM perturbations takes place for shearing rates of about an order of magnitude smaller then necessary for development of shear instability.

The remainder of the paper is organized in the following way: the general mathematical and heuristic formalism is presented in the Sec. 2. The over-reflection of GAW is studied in the Sec. 3; and the second non-adiabatic process, viz. the generation of the GAW by the EM perturbations, is studied in the Sec. 4. Finally, in Sec. 5 we conclude with a summary and a brief discussion of our results in the context of their possible importance for the angular momentum redistribution within solar-type stars and the solution of the problem of the uniform rotation of their radiative zones.

2. Linear perturbations of sheared plasma flows

For the sake of generality, we derive the basic set of equations for the case of 3D perturbations in a compressible, gravitationally stratified MHD fluid. This formalism will be presented in the next (2.1) subsection. Later on (subsection 2.2), we will restrict the consideration to the somewhat simpler case of 2D perturbations in an incompressible medium. Besides, the *Boussinesq approximation* will be used and the stationary shear flow will be considered.

2.1. General theory

In our model, the geometry of the considered problem is simplified in the following way: the equilibrium flow $\mathbf{U}$ is supposed to be plane-parallel, to be directed along the x-axis, and

to have both a horizontal (A_y) and a vertical (A_z) shear:

$$\mathbf{U} \equiv [A_y y + A_z z, 0, 0] \,. \tag{1}$$

The uniform gravity $\mathbf{g}$ is assumed to be directed along the negative direction of the z-axis:

$$\mathbf{g} \equiv [0, 0, -g] \,. \tag{2}$$

We consider a simplified model and assume that the equilibrium magnetic field $\mathbf{B_0}$ is toroidal, parallel to $\mathbf{U}$ and that it is possessing the gravity-induced vertical stratification:

$$\mathbf{B_0} \equiv [B(z),\ 0,\ 0\ ] \,. \tag{3}$$

The set of equations of one-fluid ideal magnetohydrodynamics (MHD), governing the physics of the flow is:

$$D_t\rho + \rho(\nabla \cdot \mathbf{V}) = \mathbf{0}, \tag{4}$$

$$\rho D_t \mathbf{V} = -\nabla\left(P + \frac{B^2}{8\pi}\right) + \frac{1}{4\pi}(\mathbf{B}\cdot\nabla)\mathbf{B} + \rho\mathbf{g}, \tag{5}$$

$$D_t\mathbf{B} = (\mathbf{B}\cdot\nabla)\mathbf{V} - \mathbf{B}(\nabla\cdot\mathbf{V}), \tag{6}$$

$$D_t S = 0, \tag{7}$$

while

$$\nabla \cdot \mathbf{B} = \mathbf{0}. \tag{8}$$

In (4–7) the total (convective) time derivative operator is denoted by: $D_t \equiv \partial_t + (\mathbf{V}\cdot\nabla)$.

Instantaneous values of all physical variables are decomposed into their regular (mean) and perturbational components:

$$\rho \equiv \rho_0 + \varrho, \tag{9}$$

$$P \equiv P_0 + p, \tag{10}$$

$$S \equiv S_0 + s, \tag{11c}$$

$$\mathbf{V} \equiv \mathbf{U} + \mathbf{u}, \tag{12a}$$

$$\mathbf{B} \equiv \mathbf{B}_0 + \mathbf{b}. \tag{12b}$$

It is straightforward to see that the horizontal background flow does not affect the vertical hydromagnetic equilibrium of the flowing MHD fluid, governed by the equation:

$$\partial_z\left(P_0 + \frac{B_0^2}{8\pi}\right) = -\rho_0 g. \tag{13}$$

In general the variation of the density $d\rho$ is related with variations of the pressure dP and entropy ds via the relation:

$$d\rho = \mu ds + (1/C_s^2)dp, \tag{14}$$

where:

$$\mu \equiv (\partial\rho/\partial s)_p, \tag{15}$$

$$C_s^2 \equiv (\partial P/\partial\rho)_s. \tag{16}$$

Linearized set of equations governing the evolution of perturbations within this flow can be written in the following way $[\mathcal{D}_t \equiv \partial_t + (A_y y + A_z z)\partial_x]$:

$$\mathcal{D}_t\varrho + (\partial_z\rho_0)v_z + \rho_0(\nabla\cdot\mathbf{u}) = 0, \tag{17}$$

$$\mathcal{D}_t u_x = -\mathcal{A}u_y - Au_z - \frac{1}{\rho_0}\partial_x p + \left[\frac{(\partial_z B_0)}{4\pi\rho_0}\right]b_z, \tag{18}$$

$$\mathcal{D}_t u_y = -\frac{1}{\rho_0}\partial_y p - \left(\frac{B_0}{4\pi\rho_0}\right)(\partial_x b_y - \partial_y b_x), \tag{19}$$

$$\mathcal{D}_t u_z = -\frac{1}{\rho_0}\partial_z p + \left(\frac{B_0}{4\pi\rho_0}\right)(\partial_x b_z - \partial_z b_x) - \left[\frac{(\partial_z B_0)}{4\pi\rho_0}\right]b_x - \frac{g}{\rho_0}\varrho, \tag{20}$$

$$\mathcal{D}_t s + (\partial_z s)v_z = 0, \tag{21}$$

$$\mathcal{D}_t b_y = B_0\partial_x u_y, \tag{22}$$

$$\mathcal{D}_t b_z = B_0\partial_x u_z, \tag{23}$$

$$\partial_x b_x + \partial_y b_y + \partial_z b_z = 0. \tag{24}$$

Due to the stratification some coefficients on the right hand sides of Eqs. (17-23) depend on z. But when studying the dynamics of small-scale perturbations (with $\ell_z \ll z_0$) one can consider the mean components as constant. In this case the set of first-order, partial differential equations (17-24) can be reduced to the set of ordinary differential equations (ODEs) with time-dependent coefficients if we look for solutions in the following form:

$$F(\mathbf{r},t) \equiv \hat{F}[\mathbf{k}(t),t]e^{i[(\mathbf{k}\cdot\mathbf{r})-\varphi(\mathbf{k},t)]}, \tag{25}$$

with

$$\varphi(\mathbf{k},t) \equiv \left(\mathbf{U}\cdot\int\mathbf{k}(t)dt\right), \tag{26}$$

where $\mathbf{k}(t)$ is a time-dependent wavenumber vector, determined by the following set of equations (Lagnado et al. 1984; Craik & Criminale 1986; Mahajan & Rogava 1999):

$$\mathbf{k}^{(1)} + \mathcal{S}^T \cdot \mathbf{k} = 0, \tag{27}$$

with $\mathcal{S}^T$ being the transposed *shear matrix*:

$$\mathcal{S} \equiv \begin{pmatrix} U_{x,x} & U_{x,y} & U_{x,z} \\ U_{y,x} & U_{y,y} & U_{y,z} \\ U_{z,x} & U_{z,y} & U_{z,z} \end{pmatrix}, \tag{28}$$

which, in our case has only two nonzero components: A_y and A_z:

$$\mathcal{S} \equiv \begin{pmatrix} 0 & A_y & A_z \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{29}$$

Due to Eq. (27), transversal components of the $\mathbf{k}(t)$ acquire linear time-dependence:

$$k_y(t) = k_y(0) - A_y t k_x, \tag{30a}$$

$$k_z(t) = k_z(0) - A_z t k_x. \tag{30b}$$

Therefore, we see that the wave-number vector $\mathbf{k}(t)$ of the Spatial Fourier Harmonics (SFH) varies in time, i.e. in the linear approximation there is a "drift" of the SFH in the $\mathbf{k}$-space. The physical reason of this "drift" is related to the fact that in the sheared flow the perturbations cannot have the form of a simple plane wave due to the effect of the shearing background on the wave crests.

Applying the ansatz (25) to Eqs. (17-23), we reduce the system to the following set of first-order ODEs [hereafter we are using the following notation $f^{(n)} \equiv \partial_t^n f$]:

$$d^{(1)} = i\varepsilon v_z + v_x + \mathcal{K}_y(\tau) v_y + \mathcal{K}_z(\tau) v_z, \tag{31}$$

$$v_x^{(1)} = -\mathcal{R} v_y - R v_z - \sigma^2 \mathcal{P} + i\frac{\varepsilon}{2} b_z, \tag{32}$$

$$v_y^{(1)} = -\sigma^2 \mathcal{K}_y(\tau) \mathcal{P} + b_y - \mathcal{K}_y(\tau) b_x, \tag{33}$$

$$v_z^{(1)} = -\sigma^2 \mathcal{K}_z(\tau) \mathcal{P} + b_z - \mathcal{K}_z(\tau) b_x - i\frac{\varepsilon}{2} b_x + i\alpha d, \tag{34}$$

$$e^{(1)} = v_z, \tag{35}$$

$$b_y^{(1)} = -v_y, \tag{36}$$

$$b_z^{(1)} = -v_z, \tag{37}$$

while the Maxwell equation (24) reduces to the following algebraic relation:

$$b_x + \mathcal{K}_y(\tau)b_y + \mathcal{K}_z(\tau)b_z = 0. \tag{38}$$

In these equations, the following dimensionless notation [2] is used for the constants: $R_y \equiv A_y/k_xC_A$, $R_z \equiv A_z/k_xC_A$, $\mathcal{K}_y(0) \equiv k_y(0)/k_x$, $\mathcal{K}_z(0) \equiv k_z(0)/k_x$, $\varepsilon \equiv (k_z z_0)^{-1}$, $\sigma^2 \equiv (C_s/C_A)^2$, $N^2 \equiv (N_{BV}/C_A k_x)^2$. The dimensionless variables appearing in the above set of equations are defined as: $v_j \equiv \hat{u}_j/C_A$, $b_j \equiv i\hat{b}_j/B_0$, $d \equiv i\hat{\varrho}/\rho_0$, $\mathcal{P} \equiv i\hat{p}/\rho_0 C_s^2$, $e \equiv -k_x\hat{s}/(\partial_z s_0)$, $\tau \equiv k_xC_At$, $\mathcal{K}_y(\tau) \equiv \mathcal{K}_y(0) - \mathcal{R}\tau$, $\mathcal{K}_z(\tau) \equiv \mathcal{K}_z(0) - R\tau$.

Notice that this system is *not* closed, because we have only *eight* equations for *nine* variables $(\mathbf{v}, \mathbf{b}, d, e, \mathcal{P})$. The closure of the set of equations is ensured by the thermodynamic relation that follows from Eq. (14) $[\alpha \equiv g/k_xC_s^2]$:

$$d = \mathcal{P} - i(N^2/\alpha)e. \tag{39}$$

This system of equations describes the temporal evolution of Gravito-MHD waves modified by the presence of a velocity shear in the two planes transversal to the flow direction. The full analysis of this set of equations is beyond the scope of the present paper. Instead, in the next sections, we focus our study on the relatively simple 2D and incompressible case. We will see that even in this simplified case the presence of the velocity shear brings a considerable novelty in the dynamics of perturbations.

2.2. The incompressible, 2D limit

In order to simplify the mathematical aspects of the problem, we make the following assumptions:

- We consider the 2D case, restricting the problem to the $x0z$ plan and implying:

  $$\mathcal{K}_y(0) = R_y = 0. \tag{40}$$

  Hereafter we write $R \equiv R_z (A \equiv A_z)$ In this case, Eq. (38) enables us to express the longitudinal (x-)component of the magnetic field perturbation in terms of its transversal (z-)component:

  $$b_x = -\mathcal{K}_z(\tau)b_z. \tag{41}$$

[2]The Alfvén speed $C_A^2 = B_0^2/4\pi\rho_0$ and the Brunt-Väisälä frequency $N_{BV}^2 = -g\partial_z\rho_0/\rho_0$ are defined in the usual way.

- We drop the $d^{(1)}$ term in the continuity equation, i.e. we adopt the concept of so-called *dynamic incompressibility* (Landau & Lifshitz 1959). This implies that the velocity of the flow is assumed to be small enough in comparison with the speed of sound. As a consequence, the density perturbations evoked by the pressure perturbations are negligible and instead of the Eq. (39) we will have:

$$d \simeq -i(N^2/\alpha)e. \tag{42}$$

However, since the medium is thermally conducting its density should vary also due to the temperature variation, and this effect should be (and is) taken into account.

In hydrodynamics, the second assumption is known as the *Boussinesq approximation*: in the conservation equations the terms proportional to the density perturbation are retained if, and only if, they produce a buoyancy force [3]. The Boussinesq approximation automatically implies (Landau & Lifshitz 1959; Gill 1982) that the vertical length scale of perturbations, $\ell_z \sim k_z^{-1}$, is much smaller then the characteristic stratification length scale z_0. This condition was already used for the derivation of Eqs. (31-37). In terms of our dimensionless notation, this condition transcribes into:

$$\varepsilon \ll 1. \tag{43}$$

When this condition is satisfied, the above approach enables us to study the dynamics of small-scale, low-frequency perturbations. [4]

These assumptions make the basic set of equations much simpler. Taking into account (41) and (42), we get [with $\mathcal{K}^2(\tau) \equiv 1 + \mathcal{K}_z^2(\tau)$]:

$$v_x + \mathcal{K}_z(\tau) v_z = 0, \tag{44}$$

$$v_x^{(1)} = -Rv_z - \sigma^2 \mathcal{P}, \tag{45}$$

$$v_z^{(1)} = -\sigma^2 \mathcal{K}_z(\tau)\mathcal{P} + \mathcal{K}^2(\tau) b_z + N^2 e, \tag{46}$$

$$e^{(1)} = v_z, \tag{47}$$

$$b_z^{(1)} = -v_z. \tag{48}$$

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[3]The last term in Eq. (34).

[4]for detailed analysis of the Boussinesq approximation in stratified shear flows see Didebulidze (1997) and Didebulidze et. al (2004).

From the last two equations it is apparent that there is an algebraic relation between the entropy perturbations and the magnetic field perturbations, viz.

$$e + b_z = \mathcal{I} = const(\tau). \tag{49}$$

Another, less obvious, relation between the perturbed quantities may be found if one takes time derivative of (44) and takes into account (45) and (46). The result can be used to express the pressure perturbation as a function of the other variables, i.e.

$$\sigma^2 \mathcal{P} = \frac{-2R}{\mathcal{K}^2(\tau)} v_y + \beta b_y - \frac{W^2 \beta}{\mathcal{K}^2(\tau)} e. \tag{50}$$

Having three algebraic relations, Eqs. (44), (49), and (50), for five variables, we can reduce the system to *one* second-order ODE. For instance, for the variable $b(\tau)$, defined as $b(\tau) \equiv \mathcal{K}(\tau) b_z$, we can derive the following explicit, second-order inhomogeneous ODE:

$$\frac{\mathrm{d}^2 b}{\mathrm{d}\tau^2} + \left[1 + \frac{N^2}{\mathcal{K}^2(\tau)} - \frac{R^2}{\mathcal{K}^4(\tau)}\right] b = \frac{N^2}{\mathcal{K}(\tau)} \mathcal{I}. \tag{51}$$

This equation describes the dynamics of the so-called *continuous eigenspectrum* (see, *e.g.* Balmforth & Morrison (2002) and the references therein). In order to take into consideration the perturbations belonging to the *discrete eigenspectrum* one should specify some boundary conditions with respect to the vertical spatial variable z. In the absence of an external magnetic field, it is known (Drazin & Reid 1981) that all the discrete eigenmodes are stable if the Richardson number, defined as $Ri \equiv N_{BV}^2/A^2 = N^2/R^2$, is greater then 1/4.

Equation (51) describes two important physical phenomena related to the dynamics of the continuous eigenspectrum, which will be considered in detail in the two subsequent sections, viz. (a) the (over)-reflection of the GAW and (b) the conversion of the EM into tghe GAW. It will be shown that the EM perturbations can be a very efficient source of GAW, even for relatively small shearing rates ($Ri > 1/4$).

3. Over-reflection of Gravito-Alfvén waves

In order to describe the phenomenon of the *over-reflection* of the GAW in its pure form, let us study Eq. (51) with $\mathcal{I} = 0$, yielding:

$$\frac{\mathrm{d}^2 b}{\mathrm{d}\tau^2} + \left[1 + \frac{N^2}{\mathcal{K}^2(\tau)} - \frac{R^2}{\mathcal{K}^4(\tau)}\right] b = 0. \tag{52}$$

The standard methods of the analysis of similar equations (Olver 1974) is well-known in quantum mechanics (Landau & Lifshitz 1977). The behavior of the solution is fully determined by the shear-modified frequency:

$$\omega^2(\tau) \equiv 1 + \frac{N^2}{\mathcal{K}^2(\tau)} - \frac{R^2}{\mathcal{K}^4(\tau)}. \tag{53}$$

From the mathematical point of view, the (over)-reflection is caused by the cavity of $\omega^2(\tau)$ that appears in the vicinity of the point τ^*, where $K_z(\tau^*) = 0$.

In the areas where $K_z(\tau) \gg R$, the evolution of $b(\tau)$ is adiabatic $[\mathrm{d}\omega(\tau)/\mathrm{d}\tau \ll \omega^2(\tau)]$ and, therefore, the approximate solution has the form

$$b(\tau) \approx A_+ b_+(\tau) + A_- b_-(\tau), \tag{54}$$

where

$$b_\pm(\tau) = \frac{1}{\sqrt{\omega(\tau)}} e^{\mp i \int \omega(\tau) d\tau} \tag{55}$$

are WKB solutions having positive and negative phase velocities along the x-axis, respectively, while the $A_\pm$ can be treated as the intensities of the corresponding types of perturbations.

Let us assume that $A^L_\pm$ and $A^R_\pm$ are the WKB amplitudes far on the left and right hand side from the point τ^*, respectively. It is well-known (Olver 1974) that the conservation of the Wronskian leads to the following invariant:

$$|A^L_+|^2 - |A^L_-|^2 = |A^R_+|^2 - |A^R_-|^2, \tag{56}$$

which physically corresponds to the conservation of the wave action (see, e.g., Gogoberidze et al. (2004) and references therein). If initially $A^L_- \equiv 0$, the refraction and transmission coefficients may be defined in the usual way:

$$Re = \frac{|A^R_-|^2}{|A^L_+|^2}, \quad D = \frac{|A^R_+|^2}{|A^L_+|^2}, \tag{57}$$

while from Eq. (56) we obviously have:

$$1 + Re = D. \tag{58}$$

The dependence of the transmission coefficient $D(R)$ on R, obtained through the numerical solution of Eq. (52) for different values of the Brunt-Väisälä frequency, viz. $N = 0$ (solid line), $N = 1$ (dashed line) and $N = 2$ (dash-dotted line), is presented on Fig. 1. The

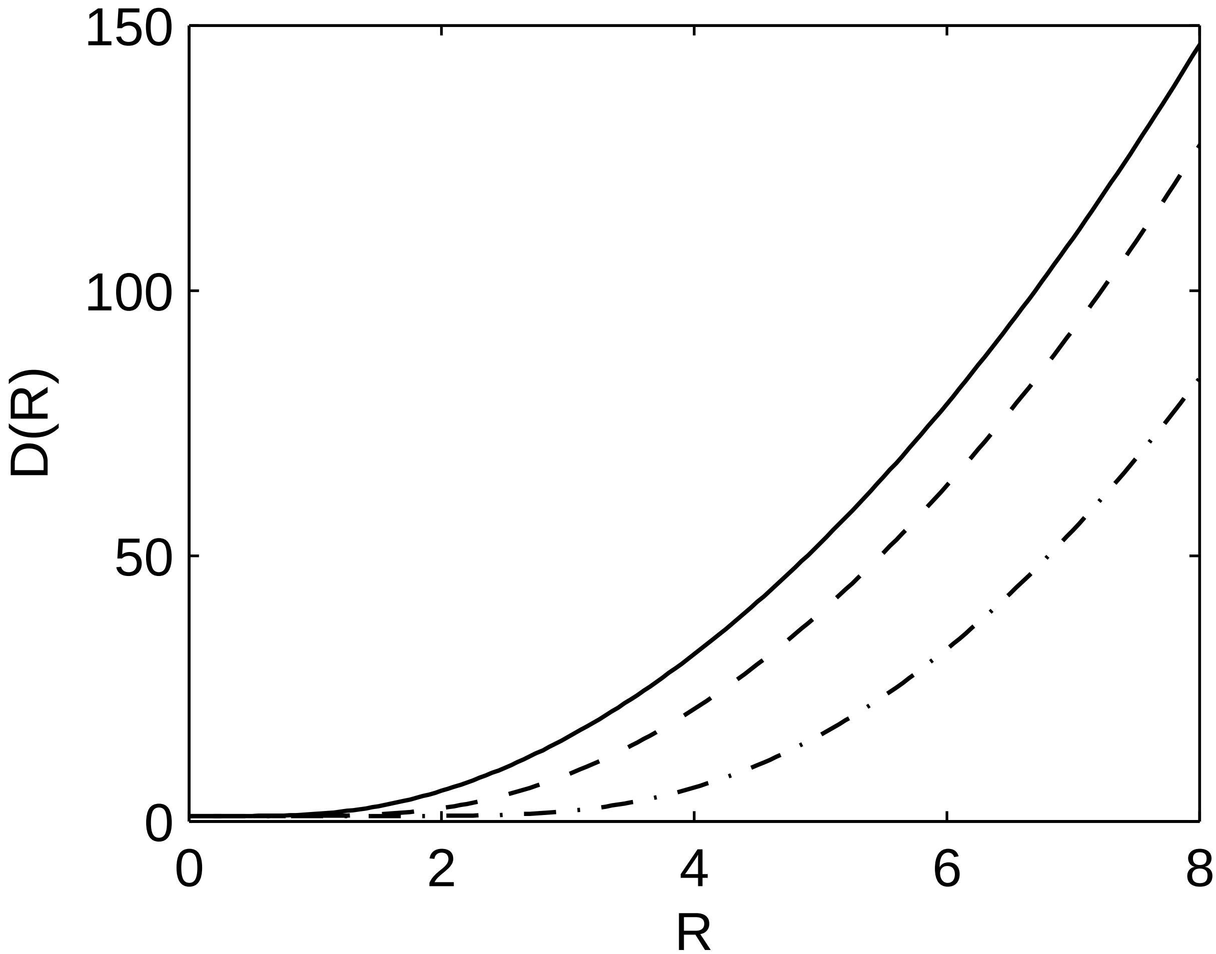

Fig. 1.— Dependence of the transmission coefficient $D(R)$ on the normalized shear parameter R for different values of the Brunt-Väisälä frequency: $N = 0$ (solid line), $N = 1$ (dashed line) and $N = 2$ (dash-dotted line).

initial conditions for the numerical solution are chosen from the WKB solutions (55) far on the left-hand side of the resonant area ($K_z(0) \gg 1, R$). The analysis leads to the conclusion that the amplification of the energy density of the perturbations is always finite, but it can become arbitrarily large with a proper increase of the shearing rate. It can be seen that the velocity shear can cause a strong amplification of the GAW, even for moderate values of the normalized shearing rates.

According to the numerical study of Eq. (52), the amplitude of the reflected wave exceeds the amplitude of the incident wave if:

$$R^2 > 2. \tag{59}$$

This phenomenon, originally discovered by Miles for acoustic waves (Miles 1957), is called *over-reflection.*

Bearing in mind that the energy of the mode is proportional to the square of its amplitude, we can easily surmise that in the course of its non-adiabatic evolution the perturbation energy is increasing. According to Eqs. (56) and (58) the ratio of the total 'post-reflection' energy to the initial one equals $Re + D > 1$. The energy increases at the expense of the mean (shear) flow energy. Consequently, over-reflection represents a quite efficient mechanism for transferring the mean flow energy to perturbations.

In the limit $R \to 0$, the reflection coefficient becomes exponentially small with respect to the parameter $-1/R$, i.e., $Re \sim \exp(-1/R)$ (Gogoberidze et al. 2004).

4. The generation of Gravito-Alfvén waves

Let us return to the analysis of the full, inhomogeneous equation (51). When the condition of the adiabatic evolution $\mathrm{d}\omega(\tau)/\mathrm{d}\tau \ll \omega^2(\tau)$ holds, then the approximate solution of Eq. (51) has the following form (Chagelishvili, Rogava & Segal 1994):

$$b(\tau) \simeq A_+ b_+(\tau) + A_- b_-(\tau) + b_E(\tau), \tag{60}$$

where the non-periodic solution $b_E(\tau)$ is given by:

$$b_E(\tau) \simeq \frac{N^2}{\mathcal{K}(\tau)} \left(1 + \frac{N^2}{\mathcal{K}^2(\tau)} - \frac{R^2}{\mathcal{K}^4(\tau)}\right)^{-1} \mathcal{I}. \tag{61}$$

and can be readily identified as the shear-modified *Entropy Mode* (EM) perturbation.

It turns out that another physically important phenomenon that takes place in the system under consideration for moderate and high shearing rates, is the generation of the

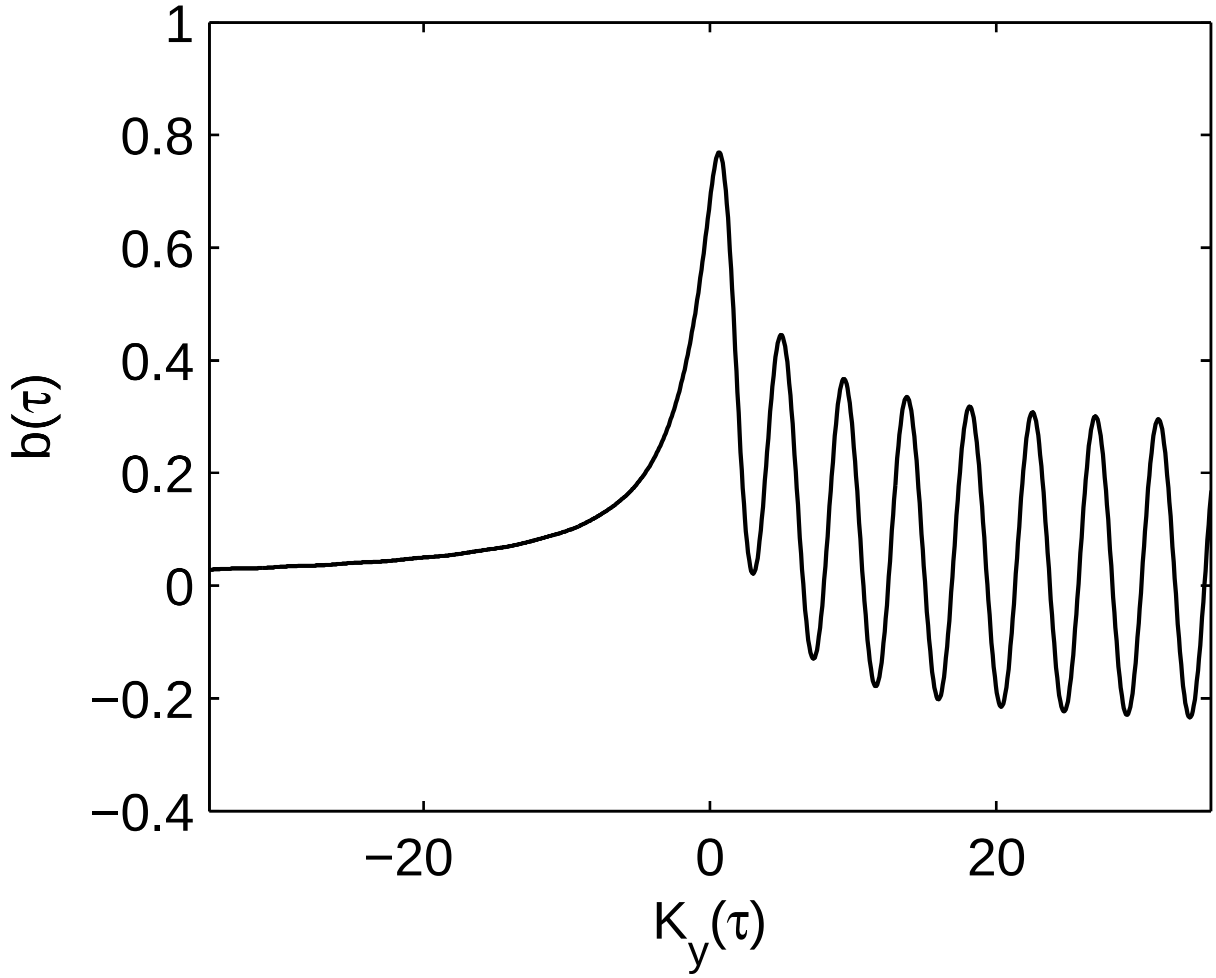

Fig. 2.— Temporal evolution of $b(\tau)$ for $N = 1$ and $R = 0.7$.

GAW by the EM perturbations. An illustrative example is given by the Fig. 2, where the numerical solution of Eq. (51) for the case when $N = 1$ and $R = 0.7$ for the variable $b(\tau)$ is shown. The initial conditions were chosen in such a way that at the beginning (at $K_z(0) = 35$) there exists only the EM perturbation. Namely, the initial conditions were found by means of Eqs. (60) and (61) with $A_\pm = 0$ and $\mathcal{I} = 1$). The figure clearly demonstrates the generation of the GAW while the EM passes through the interval of non-adiabatic evolution, situated in the vicinity of the point $\mathcal{K}_z(\tau^*) = 0$.

For the quantitative description of the wave generation process, analogously to the reflection and transmission coefficients defined in the previous section, we define the *wave generation coefficient* $G_\pm$ in the following way:

$$G_\pm = \frac{|A^L_\pm|^2}{|\mathcal{I}|^2 N^2}. \tag{62}$$

With this definition, $G_\pm$ is proportional to the ratio of the generated wave energy to the energy of incident entropy mode perturbation. From the symmetry of the problem, it is obvious that $G_+ = G_- \equiv G$.

In Figure 3, the dependence of the generation coefficient $G(R)$ on the normalized shear parameter R for different values of the Brunt-Väisälä frequency: $N = 1$ (solid line), $N = 2$ (dashed line) and $N = 3$ (dash-dotted line) is presented.

5. Discussion

In order to explain the near uniformity of the rotation profile in the radiative region of the Sun and similar regions in solar-type stars usually two different mechanisms are being proposed. The first one implies the presence of travelling IGWs generated at the base of the convective region (Charbonnel & Talon 2005), while the second one is based on the presumed (and for the Sun observationally justified) existence of the magnetic field in the radiative zone (Gough & McIntyre 1998).

It seems reasonable to assume that the above-mentioned two mechanisms are not necessarily alternatives, but could be envisaged as complementary ones. In particular, we may argue that if magnetic fields are indeed present inside the radiative zone and the Gravito-MHD waves do pass through the zone, it leads to intense energy exchange processes between the waves and the stellar plasma. This circumstance, in its turn, might lead to the more efficient redistribution of the angular momentum within these stars, establishing quasi-flat rotation profiles. The presence of the magnetic field implies that it is more reasonable to

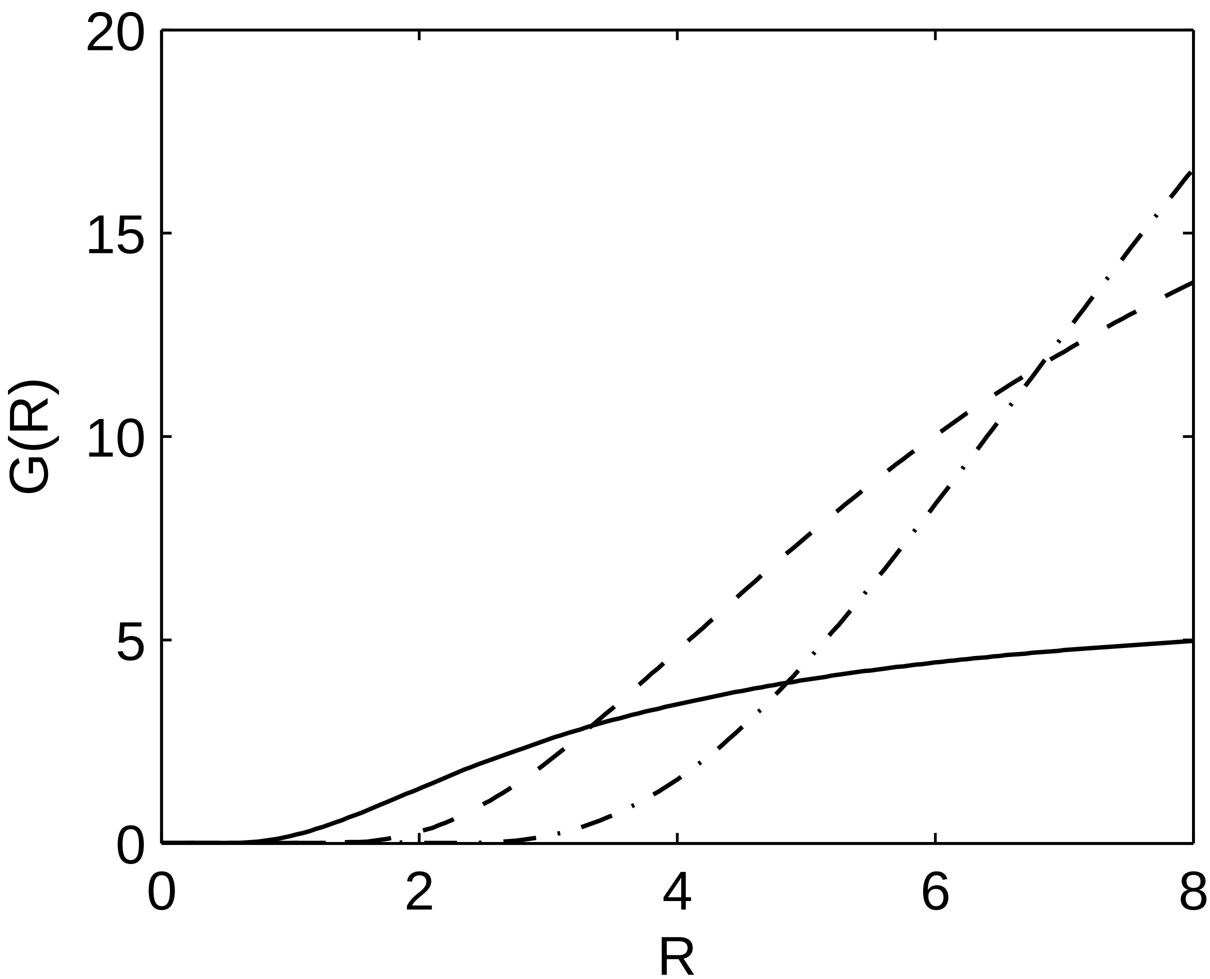

Fig. 3.— Dependence of the generation coefficient $G(R)$ on the normalized shear parameter R for different values of the Brunt-Väisälä frequency: $N = 1$ (solid line), $N = 2$ (dashed line) and $N = 3$ (dash-dotted line).

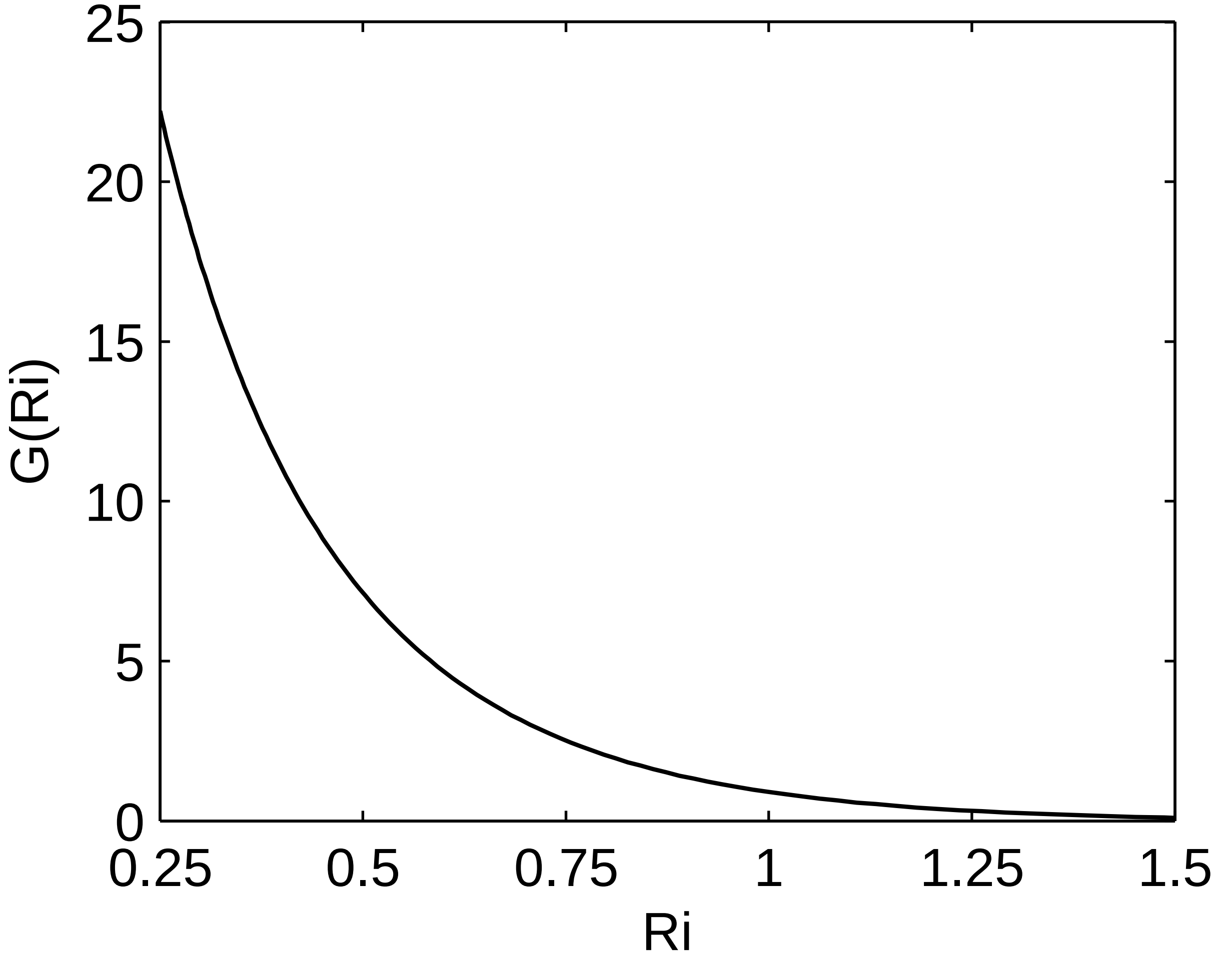

Fig. 4.— Dependence of the generation coefficient $G(Ri)$ on Richardson number Ri for $N = 4$.

speak not about purely hydrodynamical IGWs, but about Gravito-MHD waves in general, and Gravito-Alfvén waves in particular.

The evolution of the latter mode, present only in the magnetized case and absent in a purely hydrodynamic medium, is studied in the present paper. The theoretical background for studying of the evolution of the full set of Gravito-MHD waves is also developed.

From the general theory of IGWs, we know (Drazin & Reid 1981) that all the discrete eigenmodes are stable under the condition that $Ri > 1/4$. Our study shows that the generation of the GAW by the EM perturbations could provide an efficient mechanism for creating the required flux of the GAW, even for relatively low shearing rates, when the discrete eigenmodes are stable. Note, in this context, that for the case presented in Fig. 2, for instance, the Richardson number was about $Ri \approx 1.02$.

In the magnetized case, we studied the dependence of the generation coefficient $G(Ri)$ on the Richardson number Ri for the relatively weak external magnetic field ($N = 4$). The results of this study are presented in Fig. 4. The results clearly indicate that even though the generation coefficient falls rapidly with increasing Ri, the efficient generation of the GAW by the EM perturbations still takes place for $Ri < 1.25$.

Obviously, our study has a simplified nature and it only indicates at the possibly important role of *non-modal* phenomena related with GAW in the redistribution of the angular momentum and the flattening of the rotation curves in solar-type stars. For making this claim more convincing, we have to consider the compressible, three-dimensional analogue of this problem and we have to verify whether the full set of Gravito-MHD waves is as efficient (or even more efficient) in exchanging energy with the equilibrium flow. We know from previous studies that in the absence of the gravity-induced stratification among the shear-modified MHD waves, the fast magnetosonic waves are the most efficient ones in exchanging energy with the equilibrium flow (Poedts, Rogava & Mahajan 1999; Rogava, Poedts & Mahajan 2000). It is of great interest to check whether the fast Gravito-magnetosonic waves have the same quality and to determine what their contribution in the angular momentum redistribution could be. The study of the linear dynamics of all Gravito-MHD waves is currently initiated and the results will be published in a subsequent paper.

Finally, one has to remember that the non-modal approach that has been applied in the present paper, provides no information about the spatial aspects of the shear-induced processes, because the study of the Spatial Fourier Harmonics is confined to the phase space of the wave number vectors $\mathbf{k}(t)$. In order to have a clear idea about the spatial appearance of these processes, one has to study them numerically (similarly to the uniform, non-stratified flow study made by Bodo et al. (2001)) and to check how the non-modal phenomena couple

with the traditional rotational instabilities.

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